Week 10 - Monday

COMP 2230

Last time

- Finished binomial theorem
- Probability axioms
- Expected value
- Conditional probability
- Bayes' theorem

Questions?

Assignment 5

Logical warmup

- Arya and Bahar went to a reception with ten other couples
- Each person there shook hands with everyone he or she didn't know
- Later, Bahar asked each of the other 21 partygoers how many people they shook hands with and got a different answer from each one
- How many people did Arya shake hands with?

Back to Bayes

Independent events

- If events A and B are events in a sample space S, then these events are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$

- This should be clear from conditional probability
- If A and B are independent, then $P(B|A) = P(B)$

$$P(B|A) = P(B) = \frac{P(A \cap B)}{P(A)}$$

$$P(A) \cdot P(B) = P(A \cap B)$$

Graphs: Trails, Paths, and Circuits

Three-Sentence Summary

Graphs

Graphs

- A **graph** G is made up of two finite sets
  - **Vertices**: $V(G)$
  - **Edges**: $E(G)$
- Each edge is connected to either one or two vertices called its **endpoints**
- An edge with a single endpoint is called a **loop**
- Two edges with the same sets of endpoints are called **parallel**
- Edges are said to **connect** their endpoints
- Two vertices that share an edge are said to be **adjacent**
- A graph with no edges is called **empty**

The purpose of graphs

- Graphs can be used to represent connections between arbitrary things
  - Streets connecting towns
  - Links connecting computers in a network
  - Friendships between people
  - Enmities between people
  - Almost anything …

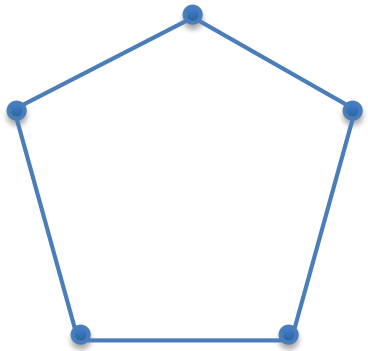
Graph representation

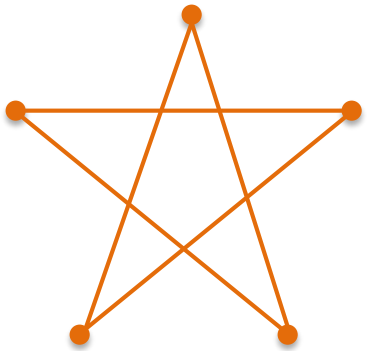
- We can represent graphs in many ways
- One is simply by listing all the vertices, all the edges, and all the vertices connected by each edge
- Let $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$
- Let $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\ e_7\}$
- Edges connect the following vertices:
- Draw the graph with the given connections

Edge	Vertices
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_1, v_3\}$
e_4	$\{v_2, v_3\}$
e_5	$\{v_5, v_6\}$
e_6	$\{v_5\}$
e_7	$\{v_6\}$

Drawing graphs

- Graphs can (generally) be drawn in many different ways
- We can label graphs to show that they are the same
- Label these two graphs to show they are the same:





Special graphs

- A **simple graph** does not have any loops or parallel edges
- Let n be a positive integer
- A **complete graph** on n vertices, written K_n, is a simple graph with n vertices such that every pair of vertices is connected by an edge
- Draw K_1, K_2, K_3, K_4, K_5
- A **complete bipartite graph on** (m, n) **vertices**, written $K_{m,n}$ is a simple graph with a set of m vertices and a disjoint set of n vertices such that:
  - There is an edge from each of the m vertices to each of the n vertices
  - There are no edges among the set of m vertices
  - There are no edges among the set of n vertices
- Draw $K_{3,2}$ and $K_{3,3}$
- A **subgraph** is a graph whose vertices and edges are a subset of another graph

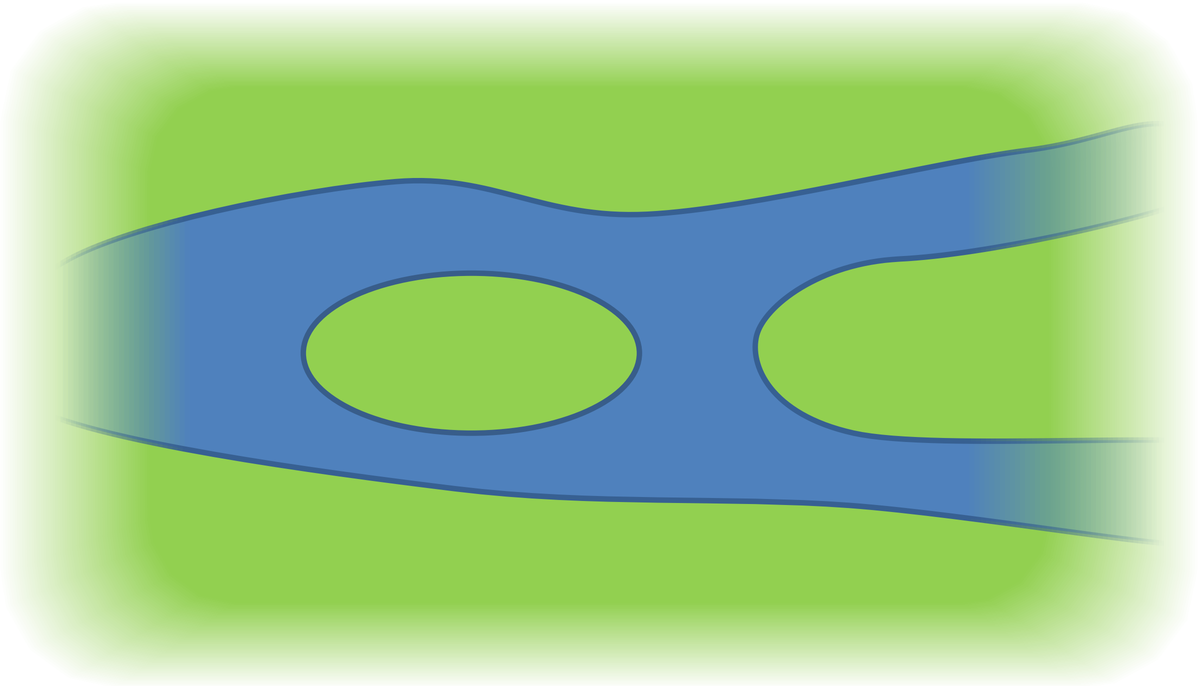
Degree

- The **degree of a vertex** is the number of edges that are incident on the vertex
- The **total degree of a graph** G is the sum of the degrees of all of its vertices
- What's the relationship between the degree of a graph and the number of edges it has?
- What's the degree of a complete graph with n vertices?
- Note that the number of vertices with odd degree must be even … why?

Paths and Circuits

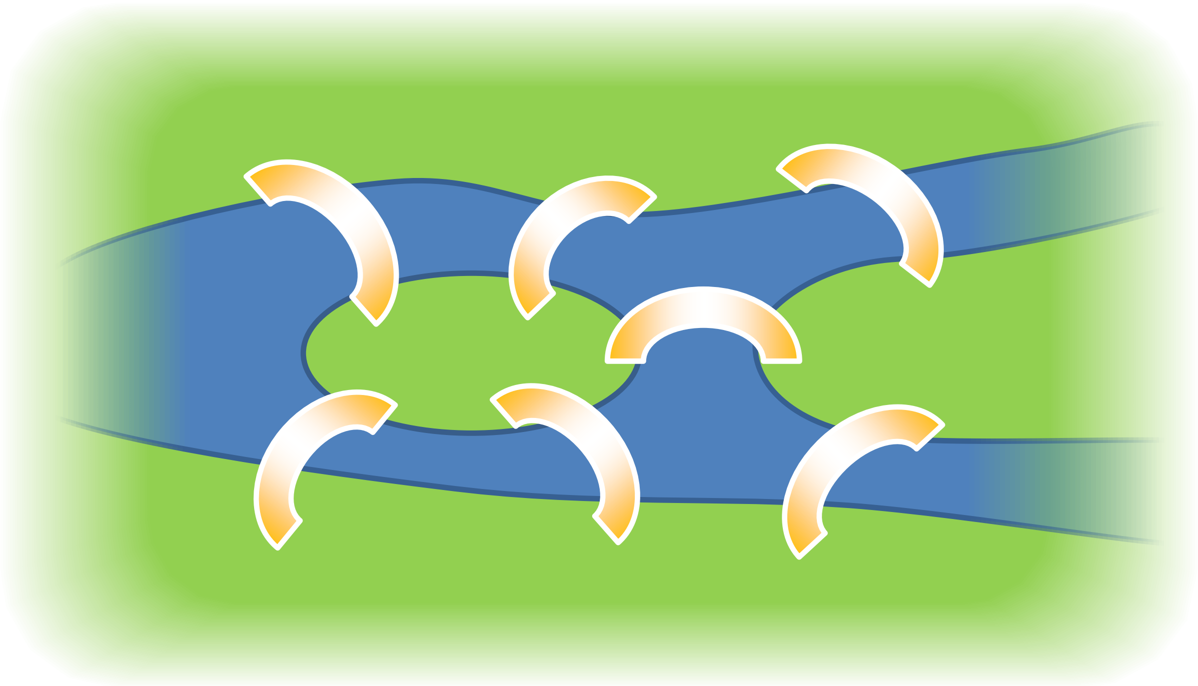
Königsberg

- Used to be Königsberg, Prussia
- Now called Kaliningrad, Russia
- On the Pregel River, including two large islands



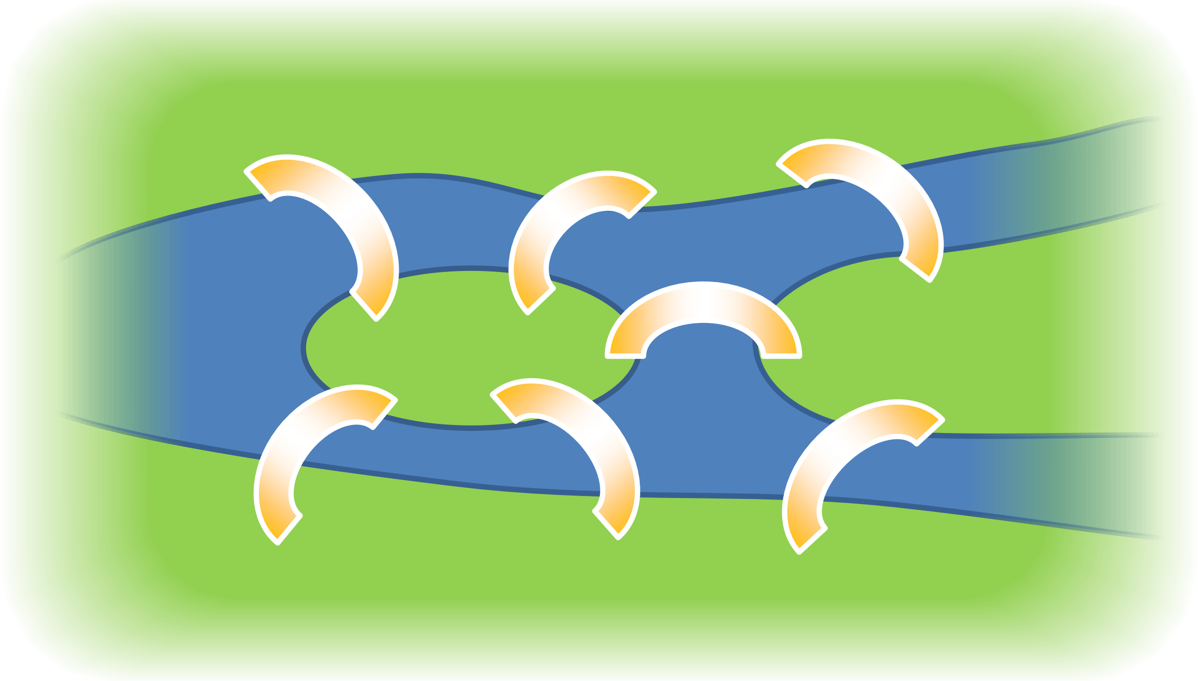
Seven Bridges of Königsberg

- In 1736, the islands were connected by seven bridges
- In modern times, there are only five



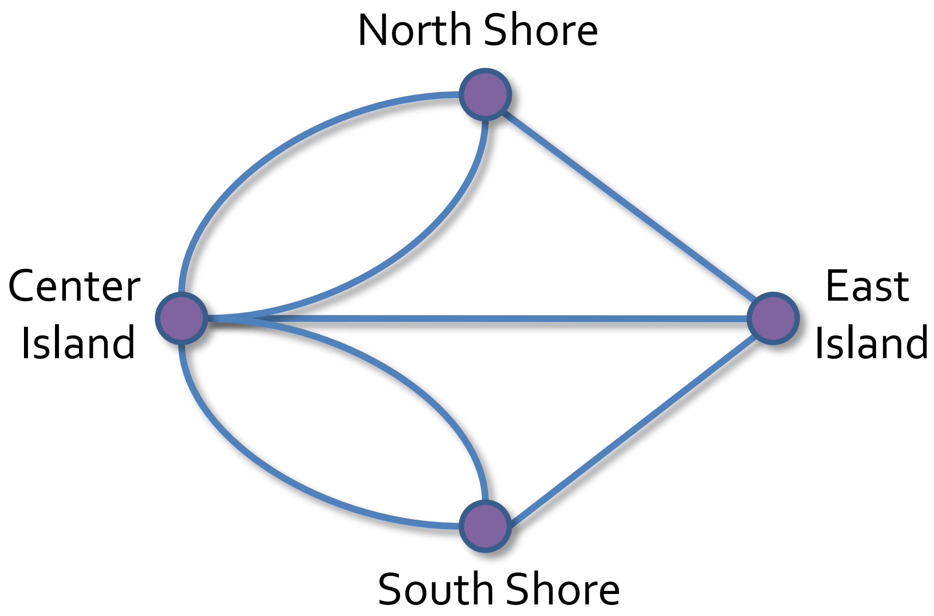
The Challenge

- After a lazy Sunday and a bit of drinking, the citizens would challenge each other to walk around the city and try to find a path which crossed each bridge exactly once



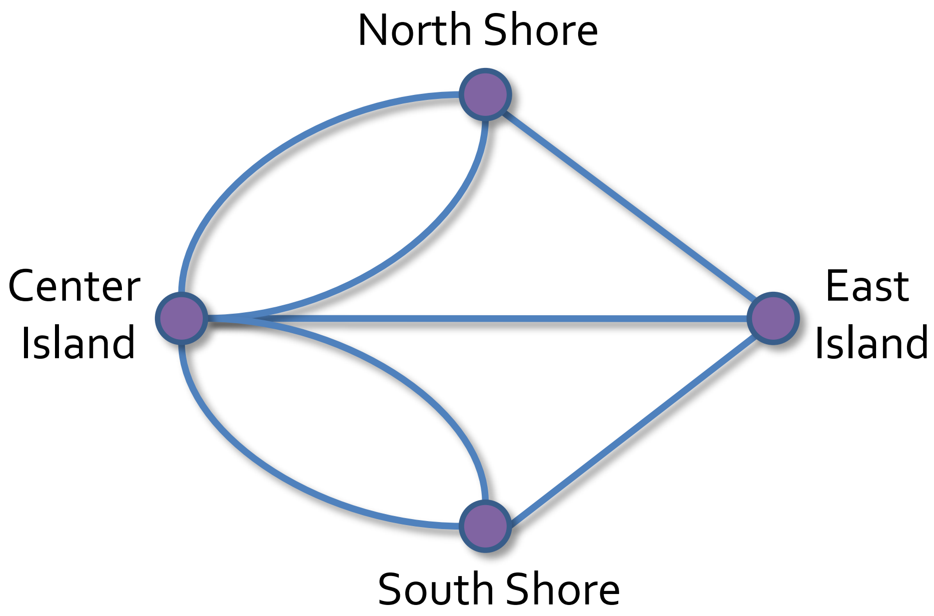
Euler's Solution

- What did Euler find?
- The same thing you did: nothing
- But, he also proved it was impossible
- Here's how:



Graph Theoretical View

- By simplifying the problem into a graph, the important features are clear
- To arrive as many times as you leave, the degrees of each node must be even (except for the starting and ending points)

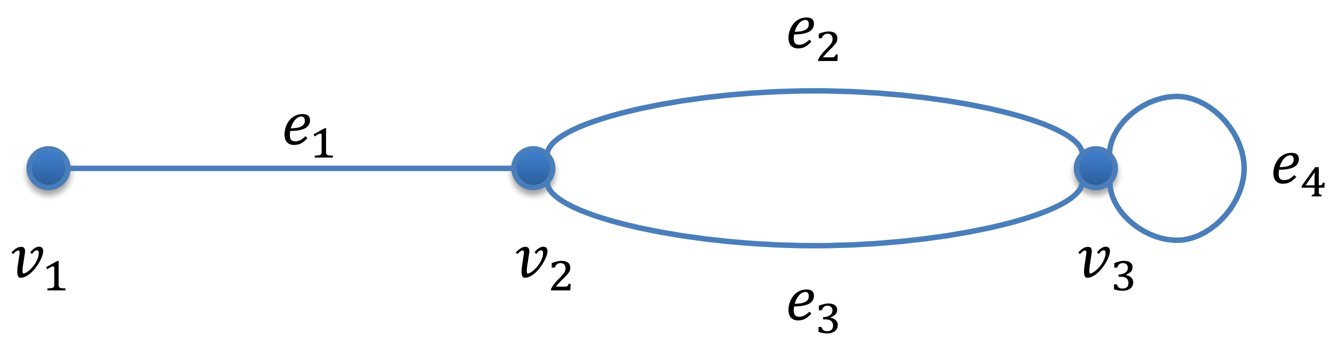


Definitions

- A **walk from** v **to** w is a finite alternating sequence of adjacent vertices and edges of G, starting at vertex v and ending at vertex w
  - A walk must begin and end at a vertex
- A **path from** v **to** w is a walk that does not contain a repeated edge
- A **simple path from** v **to** w is a path that does not contain a repeated vertex
- A **closed walk** is a walk that starts and ends at the same vertex
- A **circuit** is a closed walk that does not contain a repeated edge
- A **simple circuit** is a circuit that does not have a repeated vertex other than the first and last

Notation

- We can always pin down a walk unambiguously if we list each vertex and each edge traversed
- How would we notate a walk that starts at v_1 and ends at v_2 and visits every edge exactly once in the following graph?



- However, if a graph has no parallel edges, then a sequence of vertices uniquely determines the walk

Connectedness

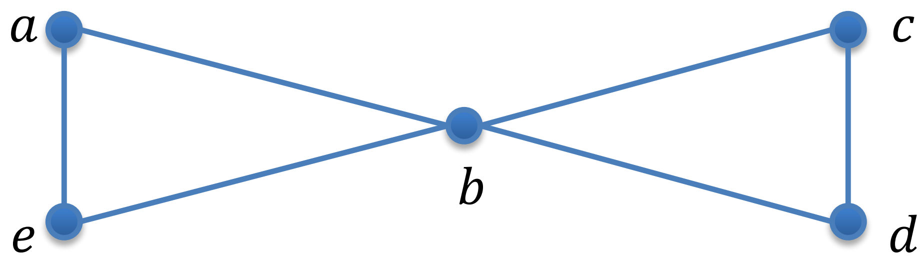
- Vertices v and w of G are connected iff there is a walk from v to w
- Graph G is connected iff all pairs of vertices v and w are connected to each other
- A graph H is a **connected component** of a graph G iff
  - H is a subgraph of G
  - H is connected
  - No connected subgraph of G has H as a subgraph and contains vertices or edges that are not in H
- A connected component is essentially a connected subgraph that cannot be any larger
- Every (non-empty) graph can be partitioned into one or more connected components

Euler circuits

- What if you want to find an Euler circuit of your own?
- If a graph is connected, non-empty, and every node in the graph has even degree, the graph has an Euler circuit
- Algorithm to find one:
  1. Pick an arbitrary starting vertex
  2. Move to an adjacent vertex and remove the edge you cross from the graph
     - Whenever you choose such a vertex, pick an edge that will not disconnected the graph
  3. If there are still uncrossed edges, go back to Step 2

Hamiltonian circuits

- An Euler circuit has to visit every edge of a graph exactly once
- A **Hamiltonian circuit** must visit every vertex of a graph exactly once (except for the first and the last)
- If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:
  - H contains every vertex of G
  - H is connected
  - H has the same number of edges as vertices
  - Every vertex of H has degree 2
- In some cases, you can use these properties to show that a graph does **not** have a Hamiltonian circuit
- In general, showing that a graph has or does not have a Hamiltonian circuit is NP-complete (widely believed to take exponential time)
- Does the following graph have a Hamiltonian circuit?



Upcoming

Next time…

- Matrix representation
- Isomorphisms

Reminders

- **Start Assignment 5**
- Read 10.2 and 10.3
  - Prepare a three-sentence summary
  - Extra credit if you get called on